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SHARP SQUARE-FUNCTION INEQUALITIES FOR CONDITIONALLY SYMMETRIC MARTINGALES

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ABSTRACT. Let f be a conditionally symmetric martingale taking values in a Hilbert space $\mathbb{H}$ and let $S(f)$ be its square function. If ν_p is the smallest positive zero of the confluent hypergeometric function and μ_p is the largest positive zero of the parabolic cylinder function of parameter p, then the following inequalities are sharp:

$$\|f\|_p \leq \nu_p \|S(f)\|_p \quad \text{if } 0 < p \leq 2,$$
$$\|f\|_p \leq \mu_p \|S(f)\|_p \quad \text{if } p \geq 3,$$
$$\nu_p \|S(f)\|_p \leq \|f\|_p \quad \text{if } p \geq 2.$$

Moreover, the constants ν_p and μ_p for the cases mentioned above are also best possible for the Marcinkiewicz-Paley inequalities for Haar functions.

1. INTRODUCTION

Let W_t, $0 \leq t < \infty$, be standard Brownian motion. It is known that there exist positive constants A_p and a_p such that for any stopping time T of W_t,

$$\|W_T\|_p \leq A_p \|T^{1/2}\|_p, \quad \text{if } 0 < p < \infty, \tag{1.1}$$

and

$$a_p \|T^{1/2}\|_p \leq \|W_T\|_p, \quad \text{if } 1 < p < \infty \quad \text{and} \quad \|T^{1/2}\|_p < \infty. \tag{1.2}$$

For the exponents $p > 1$, these follow from the inequalities of Burkholder in [3]; see, for example, Millar [11]. Burkholder and Gundy in [6] extended (1.1) to the exponents $0 < p \leq 1$. See the work of Novikov [13] for a different method and [4] for more information and related results.

Davis in [7] obtained the best possible values for the constants a_p and A_p. For $p = 2n$, n a positive integer, they are respectively ν_p and μ_p, where ν_p and μ_p are the smallest and largest positive zeros of the Hermite polynomial

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of order $2n$. When $p = 4$, this had been proven by Novikov in [12], and it is well known that the best values for a_2 and A_2 are 1. For more general p, things are little more complicated. Let ν_p be the smallest positive zero of M_p, the confluent hypergeometric function, and μ_p be the largest positive zero of D_p, the parabolic cylinder function of parameter p. We will define M_p and D_p in more detail later in §2. When $p = 2n$, both M_p and D_p become the Hermite polynomial of order $2n$. Then the best possible constants for A_p are ν_p when $0 < p \leq 2$ and μ_p for $2 \leq p < \infty$. On the other hand, the best possible constants for a_p are μ_p when $1 < p \leq 2$ and ν_p when $2 \leq p < \infty$.

Brownian motion is a continuous time martingale. In the analogues of (1.1) and (1.2) for discrete time martingales, less is known about the best possible values for the constants a_p and A_p. Recall that $f = (f_1, f_2, \dots, f_n, \dots)$, a sequence of real integrable functions on a probability space $(\Omega, \mathscr{A}, P)$, is a real martingale if d_{n+1} is orthogonal to $\varphi(d_1, \dots, d_n)$ for all real bounded continuous functions φ on $\mathbb{R}^n$ and all $n \geq 1$, where $(d_1, d_2, \dots, d_n, \dots)$ is the difference sequence of f: $f_n = \sum_{k=1}^n d_k$. This is equivalent to

$$E(f_{n+1}|f_1, f_2, \dots, f_n) = f_n \quad \text{a.e. for all } n \geq 1.$$

Let $S_n(f) = (\sum_{k=1}^n |d_k|^2)^{1/2}$. We also use the notations $\|f\|_p$ and $S(f)$ standing for $\sup_n (E|f_n|^p)^{1/p}$ and $\lim_{n\to\infty} S_n(f)$ respectively. The function $S(f)$ is called the square function of f.

If $\mathbb{H}$ is a Hilbert space, we can define an $\mathbb{H}$-valued martingale in a similar way: The integral of the product of the $\mathbb{H}$-valued strongly integrable function d_{n+1} with the scalar-valued function $\varphi(d_1, d_2, \dots, d_n)$, where φ is bounded and continuous on $\mathbb{H}^n$, is equal to the origin of $\mathbb{H}$. If the norm of $\mathbb{H}$ is denoted by $|\cdot|$, then $S_n(f)$, $S(f)$, and $\|f\|_p$ are defined as above.

Let $1 < p < \infty$. In [3], Burkholder showed that there exist positive constants b_p and B_p such that for all real-valued martingales f,

$$b_p\|S(f)\|_p \leq \|f\|_p \leq B_p\|S(f)\|_p. \tag{1.3}$$

Recently Burkholder [5] proved the following extension of (1.3) and, at the same time, obtained some information about the best constants.

Theorem A. *If* $1 < p < \infty$, *then, for any* $\mathbb{H}$*-valued martingale* f,

$$(p^* - 1)^{-1}\|S(f)\|_p \leq \|f\|_p \leq (p^* - 1)\|S(f)\|_p$$

where $p^* = \max(p, \frac{p}{p-1})$. *In particular,*

$$\|f\|_p \leq (p-1)\|S(f)\|_p \quad \text{if } p \geq 2, \tag{1.4}$$

and

$$(p-1)\|S(f)\|_p \leq \|f\|_p \quad \text{if } 1 < p \leq 2. \tag{1.5}$$

Moreover, the constants in (1.4) *and* (1.5) *are best possible.*

Pittenger [15] proved part of (1.4): the special case in which $p \geq 3$ and $\mathbb{H} = \mathbb{R}$. His proof can be modified to carry over to any Hilbert space, but cannot be modified to carry over to $2 < p \leq 3$ even for $\mathbb{H} = \mathbb{R}$. The best possible constants are unknown for the cases not covered by (1.4) and (1.5).

Here we consider a class of special martingales: the class of conditionally symmetric martingales. A martingale $f = (f_1, f_2, \dots)$ is conditionally symmetric if d_{n+1} and $-d_{n+1}$ have the same conditional distribution given $d_1, \dots, d_n$. To be precise, for any positive integer n and any two bounded real continuous functions τ and χ on $\mathbb{H}$ and $\mathbb{H}^n$ respectively, the integral of the product of $\tau(d_{n+1})$ with $\chi(d_1, \dots, d_n)$ is the same as that of $\tau(-d_{n+1})$ with $\chi(d_1, \dots, d_n)$. In the real case, this is equivalent to $P(d_{n+1} > a | d_1, \dots, d_n) = P(d_{n+1} < -a | d_1, \dots, d_n)$ a.e. for each positive integer n and each positive real number a. For example, let $\varphi_1, \varphi_2, \dots$ be the complete orthonormal system of Haar functions on $[0, 1]$, and let $\lambda_1, \lambda_2, \dots$ be elements of a Hilbert space $\mathbb{H}$. Then

$$f_n = \sum_{k=1}^{n} \lambda_k \cdot \varphi_k$$

defines a conditionally symmetric martingale $f = (f_1, f_2, \dots)$. In fact, any dyadic martingale is conditionally symmetric.

For $\lambda_i \in \mathbb{R}$ and the exponents $p > 1$, Marcinkiewicz [10] proved (1.3) in the Haar case by using Paley's [14] work which gave an equivalent Walsh series form. Burkholder and Gundy in [6] proved the right-hand side of (1.3) for f in the Haar case with real λ_i and exponents $0 < p \leq 1$. Davis [7] found the best possible constants B_p in (1.3) when $0 < p \leq 2$ and b_p in (1.3) when $2 \leq p < \infty$ for real conditionally symmetric martingales. They are the same as those found for A_p and a_p. He used Skorohod embedding but this does not work for $\mathbb{H}$-valued martingales.

In this paper we will find best possible constants of the right-hand side of (1.3) when $0 < p \leq 2$ and $p \geq 3$ and those of the left-hand side when $p \geq 2$ for Hilbert-space-valued conditionally symmetric martingales. We will show the following theorem:

Theorem 1. *Let f be an $\mathbb{H}$-valued conditionally symmetric martingale. Then*

$$\|f\|_p \leq \nu_p \|S(f)\|_p \quad \text{if } 0 < p \leq 2, \tag{1.6}$$

$$\|f\|_p \leq \mu_p \|S(f)\|_p \quad \text{if } p \geq 3, \tag{1.7}$$

and

$$\nu_p \|S(f)\|_p \leq \|f\|_p \quad \text{if } p \geq 2. \tag{1.8}$$

Moreover, the constants are best possible.

By the standard approximation argument we can assume that the sequence (f_n) consists of simple functions and that $\mathbb{H} = \mathbb{R}^N$ for some positive integer N. Hence, Theorem 1 is implied by the following:

Theorem 1′. *Let $f = (f_1, f_2, \dots)$ be a Hilbert-space-valued conditionally symmetric martingale of simple functions. Then*

$$\|f\|_p \le \nu_p \|S(f)\|_p \quad \text{if } 0 < p \le 2, \tag{1.6$'$}$$

$$\|f\|_p \le \mu_p \|S(f)\|_p \quad \text{if } p \ge 3, \tag{1.7$'$}$$

and

$$\nu_p \|S(f)\|_p \le \|f\|_p \quad \text{if } p \ge 2. \tag{1.8$'$}$$

Moreover, the constants are best possible.

The constants are also the best in the Haar case. This gives some new information about Marcinkiewicz-Paley inequalities.

Inequalities $(1.6)'$–$(1.8)'$ are equivalent to: For any $n \ge 1$,

$$\|f_n\|_p \le \nu_p \|S_n(f)\|_p \quad \text{if } 0 < p \le 2, \tag{1.9}$$

$$\|f_n\|_p \le \mu_p \|S_n(f)\|_p \quad \text{if } p \ge 3, \tag{1.10}$$

and

$$\nu_p \|S_n(f)\|_p \le \|f_n\|_p \quad \text{if } p \ge 2. \tag{1.11}$$

The inequalities $(1.6)'$–$(1.8)'$ imply (1.9)–(1.11) since for any $n \ge 1$, $(f_1, f_2, \dots, f_n, 0, 0, \dots)$ is a conditionally symmetric martingale. On the other hand, by taking $n \to \infty$, (1.9)–(1.11) imply $(1.6)'$–$(1.8)'$.

We also discuss what we know for the other cases, for the exponents not mentioned above.

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2. Confluent hypergeometric functions and parabolic cylinder functions

The confluent hypergeometric function M_p is closely related to Kummer's function $M(a, b, z)$, which is a solution of the differential equation

$$zw''(z) + (b - z)w'(z) - aw(z) = 0. \tag{2.1}$$

The explicit form of $M(a, b, z)$ is

$$M(a, b, z) = 1 + \frac{a \cdot z}{b} + \frac{(a)_2 \cdot z^2}{(b)_2 \cdot 2!} + \cdots + \frac{(a)_n \cdot z^n}{(b)_n \cdot n!} + \cdots \tag{2.2}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$. Let $M_p(x) = M(-\frac{p}{2}, \frac{1}{2}, \frac{x^2}{2})$, the function mentioned in §1. If $p = 2n$, then M_p is a constant multiple of the

Hermite polynomial of order $2n$ where the constant depends on n. By (2.1) and (2.2), the function M_p satisfies

$$U''(x) - xU'(x) + pU(x) = 0 \tag{2.3}$$

and

$$U(0) = 1, \qquad U'(0) = 0. \tag{2.4}$$

Differentiating (2.3) twice, we see that M_p'' satisfies

$$U''(x) - xU'(x) + (p-2)U(x) = 0. \tag{2.5}$$

From (2.2) it is clear that $M_p''(0) = -p$ and $M_p^{(3)}(0) = 0$. Thus, by the uniqueness of the solution of the differential equation (2.3) with initial conditions (2.4), we have

$$M_p''(x) = -pM_{p-2}(x). \tag{2.6}$$

As in §1, we denote the smallest positive zero of M_p by ν_p. Let $\nu_p = \infty$ if no such zero exists.

Parabolic cylinder functions are related to the confluent hypergeometric functions. They are solutions of the differential equation

$$Y''(x) + (ax^2 + bx + c)Y(x) = 0.$$

Here we consider the solutions of the special case

$$Y''(x) - (\tfrac{1}{4}x^2 - p - \tfrac{1}{2})Y(x) = 0.$$

Two linearly independent solutions are

$$y_1(x) = e^{-x^2/4} M(-\tfrac{p}{2}, \tfrac{1}{2}, \tfrac{x^2}{2})$$

and

$$y_2(x) = xe^{-x^2/4} M(-\tfrac{p}{2} + \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{x^2}{2}).$$

The parabolic cylinder function D_p, which is also known as Whittaker's function, is defined by

$$D_p(x) = Y_1(x)\cos\tfrac{p}{2}\pi + Y_2(x)\sin\tfrac{p}{2}\pi,$$

where

$$Y_1(x) = (2^{p/2}/\sqrt{\pi})\Gamma((p+1)/2)y_1(x)$$

and

$$Y_2(x) = (2^{(p+1)/2}/\sqrt{\pi})\Gamma((p+2)/2)y_2(x).$$

We shall sometimes write $U(-p-\tfrac{1}{2}, x)$ for $D_p(x)$. Let $h_p(x) = e^{x^2/4}D_p(x)$. An easy calculation yields that h_p satisfies (2.3).

From 19.6.1 and 19.8.1 of [1], we have

$$h_p(x) \sim x^p \left\{1 - \frac{p(p-1)}{2x^2} + o(x^{-3})\right\} \quad \text{as } x \to \infty, \tag{2.7}$$

and

$$\begin{aligned} h_p'(x) &= \tfrac{x}{2}e^{x^2/4}U(-p-\tfrac12, x) + e^{x^2/4}U'(-p-\tfrac12, x) \\ &= \tfrac{x}{2}e^{x^2/4}U(-p-\tfrac12, x) + e^{x^2/4}\{-\tfrac{x}{2}U(-p-\tfrac12, x) + pU(-p+\tfrac12, x)\} \\ &= ph_{p-1}(x). \end{aligned}$$

Let μ_p be the largest positive zero of h_p. If h_p has no zero or no positive zero, let $\mu_p = 0$.

3. An outline of the proofs

Recall that ν_p and μ_p are, respectively, the smallest positive zero of M_p, and the largest positive zero of D_p. We define functions v_p, V_p, $\overline{v}_p$, and $\overline{V}_p$ on $\mathbb{H}$ by

$$\begin{aligned} v_p(x) &= |x|^p - \mu_p^p \quad \text{if } p \geq 2, \\ &= \mu_p^p - |x|^p \quad \text{if } 1 \leq p < 2, \end{aligned}$$

$$\begin{aligned} V_p(x, t) &= t^{p/2} v_p(x/\sqrt{t}) \quad \text{if } t > 0, \\ &= \operatorname{sgn}(p-2)|x|^p \quad \text{if } t = 0 \end{aligned}$$

(here $\operatorname{sgn}(0) = 1$), and

$$\begin{aligned} \overline{v}_p(x) &= |x|^p - \nu_p^p \quad \text{if } 0 < p < 2, \\ &= \nu_p^p - |x|^p \quad \text{if } p \geq 2, \end{aligned}$$

$$\begin{aligned} \overline{V}_p(x, t) &= t^{p/2} \overline{v}_p(x/\sqrt{t}) \quad \text{if } t > 0, \\ &= -\operatorname{sgn}(p-2)|x|^p \quad \text{if } t = 0. \end{aligned}$$

Let $f = (f_1, f_2, \dots)$ be a real conditionally symmetric martingale of simple functions. Then for $0 < p < 2$,

$$\|f_n\|_p^p - \nu_p^p \|S_n(f)\|_p^p = E(|f_n|^p - \nu_p^p S_n(f)^p) = E\overline{V}_p(f_n, S_n^2(f)).$$

So $\|f_n\|_p \leq \nu_p \|S_n(f)\|_p$ is equivalent to $E\overline{V}_p(f_n, S_n^2(f)) \leq 0$ when $0 < p < 2$. Similarly, when $p \geq 2$, $\nu_p \|f_n\|_p \leq \|S_n(f)\|_p$ is equivalent to $E\overline{V}_p(f_n, S_n^2(f)) \leq 0$; and when $p \geq 3$, $\|f_n\|_p \leq \mu_p \|S_n(f)\|_p$ is equivalent to $EV_p(f_n, S_n^2(f)) \leq 0$. Thus (1.9)–(1.11) are equivalent to

(3.1) $$EV_p(f_n, S_n^2(f)) \leq 0 \quad \text{for } p \geq 3,$$

and

(3.2) $$E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \quad \text{for } p > 0.$$

Our method is to find functions $U_p(x, t)$ on $\mathbb{H} \times [0, \infty)$ for $p \geq 3$ and $\overline{U}_p(x, t)$ on $\mathbb{H} \times [0, \infty)$ for $p > 0$ such that

$$V_p(x, t) \leq U_p(x, t), \tag{3.3}$$

$$\tfrac{1}{2}\{U_p(x+a, t+|a|^2) + U_p(x-a, t+|a|^2)\} - U_p(x, t) \leq 0 \tag{3.4}$$

for $p \geq 3$ and all $x, a \in \mathbb{H}$, $t \in [0, \infty)$,

$$\overline{V}_p(x, t) \leq \overline{U}_p(x, t), \tag{3.5}$$

$$\tfrac{1}{2}\{\overline{U}_p(x+a, t+|a|^2) + \overline{U}_p(x-a, t+|a|^2)\} - \overline{U}_p(x, t) \leq 0 \tag{3.6}$$

for $p > 0$ and all $x, a \in \mathbb{H}$, $t \in [0, \infty)$; and

$$U_p(a, |a|^2) \leq 0 \quad \text{and} \quad \overline{U}_p(a, |a|^2) \leq 0 \quad \text{for all } a \in \mathbb{H}. \tag{3.7}$$

We call (3.3) and (3.5) *the majorization properties*, (3.4) and (3.6) *the averaging properties*, and (3.7) *the negativity property.*

Once such functions are known, then for a conditionally symmetric martingale $f = (f_1, f_2, \dots)$ with difference sequence $(d_1, d_2, \dots)$ and with each f_n simple,

$$EV_p(f_n, S_n^2(f)) \leq EU_p(f_n, S_n^2(f)), \tag{3.8}$$

$$E\overline{V}_p(f_n, S_n^2(f)) \leq E\overline{U}_p(f_n, S_n^2(f)), \tag{3.9}$$

from (3.3) and (3.5). For $m \geq 1$, on $\mathscr{D}_m = \{d_1 = \mathscr{d}_1, d_2 = \mathscr{d}_2, \dots, d_m = \mathscr{d}_m\}$, let $f_m = \sum_{i=1}^m \mathscr{d}_i$, $S_m(f) = (\sum_{i=1}^m \mathscr{d}_i^2)^{1/2}$. By conditional symmetry and (3.4),

$$\begin{aligned}
\int_{\mathscr{D}_m} U_p(f_{m+1}, S_{m+1}^2(f)) &= \int_{\mathscr{D}_m} U_p(f_m + d_{m+1}, S_m^2(f) + |d_{m+1}|^2) \\
&= \frac{1}{2}\sum_a \Bigg(\int_{\{\mathscr{D}_m, d_{m+1}=a\}} U_p(f_m + a, S_m^2(f) + |a|^2) \\
&\qquad\qquad + \int_{\{\mathscr{D}_m, d_{m+1}=-a\}} U_p(f_m - a, S_m^2(f) + |a|^2)\Bigg) \\
&= \frac{1}{2}\sum_a (P(\mathscr{D}_m, d_{m+1} = a) U_p(f_m + a, S_m^2(f) + |a|^2) \\
&\quad + P(\mathscr{D}_m, d_{m+1} = -a) U_p(f_m - a, S_m^2(f) + |a|^2)) \\
&= \frac{1}{2}\sum_a P(\mathscr{D}_m, d_{m+1} = a)(U_p(f_m + a, S_m^2(f) + |a|^2) \\
&\qquad\qquad + U_p(f_m - a, S_m^2(f) + |a|^2)) \\
&\leq \sum_a P(\mathscr{D}_m, d_{m+1} = a) U_p(f_m, S_m^2(f)) \\
&= U_p(f_m, S_m^2(f)) \cdot P(\mathscr{D}_m) = \int_{\mathscr{D}_m} U_p(f_m, S_m^2(f)).
\end{aligned}$$

Consequently, for $m \geq 1$,

$$EU_p(f_{m+1}, S_{m+1}^2(f)) \leq EU_p(f_m, S_m^2(f)).$$

Thus for $m \geq 1$,

$$EU_p(f_m, S_m^2(f)) \leq EU_p(f_1, S_1^2(f)) = EU_p(f_1, |f_1|^2). \tag{3.10}$$

But $EU_p(f_1, |f_1|^2) \leq 0$ by (3.7), so combining (3.8) and (3.10), we have (3.1). Similarly (3.2) will follow if $\overline{U}_p(x, t)$ is known.

Note. When $p > 0$, if we only find $\overline{U}_p(x, t)$ defined on $\mathscr{T} = \{(x, t): x \in \mathscr{H}, t > 0\}$ satisfying

$$\overline{V}_p(x, t) \leq \overline{U}_p(x, t), \tag{3.5$'$}$$

(3.6)$'$

$$\tfrac{1}{2}\{\overline{U}_p(x + a, t + |a|^2) + \overline{U}_p(x - a, t + |a|^2)\} - \overline{U}_p(x, t) \leq 0 \quad \text{for } a \in \mathbb{H},$$

and

$$\overline{U}_p(a, |a|^2) \leq 0 \quad \text{for all } a \in \mathbb{H}\backslash\{0\}, \tag{3.7$'$}$$

then (3.2) still holds. The reason is the following.

Extend $\overline{U}_p(x, t)$ to $\mathscr{T} \cup \{(0, 0)\}$ by defining $\overline{U}_p(0, 0) = 0$. Since $S_n(f) = 0$ implies $f_n = 0$ for all $n \geq 1$, in order to have

$$E\overline{V}_p(f_n, S_n^2(f)) \leq E\overline{U}_p(f_n, S_n^2(f)),$$

$$E\overline{V}_p(f_{n+1}, S_{n+1}^2(f)) \leq E\overline{U}_p(f_n, S_n^2(f)),$$

and

$$E\overline{U}_p(f_1, S_1^2(f)) = E\overline{U}_p(f_1, |f_1|^2) \leq 0,$$

we need only

$$\overline{V}_p(x, t) \leq \overline{U}_p(x, t), \tag{3.11}$$

$$\tfrac{1}{2}\{\overline{U}_p(x + a, t + |a|^2) + \overline{U}_p(x - a, t + |a|^2)\} - \overline{U}_p(x, t) \leq 0, \tag{3.12}$$

for all $a \in \mathbb{H}$, $(x, t) \in \mathscr{T} \cup \{(0, 0)\}$, and

$$\overline{U}_p(a, |a|^2) \leq 0 \quad \text{for all } a \in \mathbb{H}. \tag{3.13}$$

(3.11) and (3.13) follow clearly from the definition of $\overline{U}_p$ and (3.5)$'$.

If $t > 0$, (3.12) follows from (3.6)$'$. If $t = 0$, then (3.12) becomes $\overline{U}_p(a, |a|^2) \leq 0$ which follows from (3.13). Thus (3.5)$'$–(3.7)$'$ will ensure (3.2).

4. The existence of the function $U_p(x, t)$: Real case

Recall from §2 that $h_p(x) = e^{x^2/4}D_p(x) = e^{x^2/4}U(-p - \frac{1}{2}, x)$ satisfies the differential equation

$$U''(x) - xU'(x) + pU(x) = 0. \tag{4.1}$$

Moreover

$$h_p(x) \sim x^p \left\{1 - \frac{p(p-1)}{2x^2} + o(x^{-3})\right\} \quad \text{as } x \to \infty, \tag{4.2}$$

and

$$h_p'(x) = p h_{p-1}(x). \tag{4.3}$$

As in §2, let μ_p be the largest positive zero of h_p. If h_p has no zero or no positive zero, let $\mu_p = 0$.

As we will show in Lemma 5.1 of §5, for $p \geq 1$, $h_p(\mu_p) = 0$. Since $h_p(\mu_p) = 0$ and h_p is not identically zero, $h_p'(\mu_p) \neq 0$ from the uniqueness of the solution of (4.1). Thus it is meaningful to let

$$\alpha_p = v_p'(\mu_p)/h_p'(\mu_p) \quad \text{and} \quad w_p(x) = \alpha_p h_p(|x|).$$

We define for $x \in \mathbb{R}$

$$\begin{aligned} u_p(x) &= v_p(x) \quad \text{if } 0 \leq |x| \leq \mu_p, \\ &= w_p(x) \quad \text{if } \mu_p < |x| < \infty, \end{aligned}$$

and

$$\begin{aligned} U_p(x, t) &= t^{p/2} u_p(x/\sqrt{t}) \quad \text{if } t > 0, \\ &= \alpha_p |x|^p \qquad\qquad \text{if } t = 0. \end{aligned}$$

If $x \neq 0$, then, by (4.2), $V_p(x, \cdot)$ and $U_p(x, \cdot)$ are continuous functions on $[0, \infty)$, where $V_p(x, \cdot)$ is defined in §3.

In the following several sections, we shall show that

$$U_p(x, t) \geq V_p(x, t) \quad \text{for } p > 1,\ t \geq 0,\ x \in \mathbb{R}, \tag{4.4}$$

$$\tfrac{1}{2}\{U_p(x+a, t+a^2) + U_p(x-a, t+a^2)\} - U_p(x, t) \leq 0 \tag{4.5}$$

for $p \geq 3$, $t \geq 0$, $x \in \mathbb{R}$, $a \in \mathbb{R}$, and

$$U_p(a, a^2) \leq 0 \quad \text{for } p \geq 3 \text{ and } a \in \mathbb{R}. \tag{4.6}$$

The proof of (4.4)–(4.6) will complete the proof of the inequality (1.7) of Theorem 1.

We also show why (4.5) is not true for the exponents $1 < p < 2$ and $2 < p < 3$.

As before, we call (4.4) *the majorization property*, (4.5) *the averaging property*, and (4.6) *the negativity property*.

5. Proof of the majorization property

By the definition of $U_p(x, t)$ and $V_p(x, t)$ as well as the continuity, (4.4) is equivalent to

$$w_p(x) \geq v_p(x) \quad \text{for all } x \geq \mu_p. \tag{5.1}$$

In fact, an even stronger property holds:

$$w_p(x) \geq v_p(x) \quad \text{for all } x \geq 0. \tag{5.2}$$

See Wang [17] for the proof.

First we show some lemmas.

Lemma 5.1. *$\mu_p \geq \mu_q$ and $h_p(\mu_p) = 0$ for $p \geq q \geq 1$.*

Proof. Let W_t be standard Brownian motion. Consider the stopping time defined by

$$S_a = \inf\{t > 0 \colon W_t = a\sqrt{t} - 1\}, \qquad a > 0.$$

Novikov proved in [12] that S_a satisfies

$$ES_a^p < \infty \quad \text{if } a > \mu_{2p} \tag{5.3}$$

and

$$ES_{\mu_{2p}}^p = \infty \tag{5.4}$$

for $p > \frac{1}{2}$. Thus if there exist p and q such that $p > q > 1$ and $\mu_p < \mu_q$, then by (5.3) $\|S_{\mu_q}^{1/2}\|_p < \infty$. The Liapounov inequality implies that $\|S_{\mu_q}^{1/2}\|_q \leq \|S_{\mu_q}^{1/2}\|_p < \infty$, which is contrary to (5.4). Thus $\mu_p \geq \mu_q$ if $p > q > 1$.

Note that $\mu_1 = h_1(\mu_1) = 0$ since $h_1(x)$ is a constant multiplying x. So if we can show $\mu_p > 0$ for $1 < p < 2$, then the lemma is proven.

By 19.3.3 of [1],

$$h_p(0) = (2^{p/2}/\sqrt{\pi})\Gamma((p+1)/2)\cos(p\pi/2).$$

Hence $h_p(0) < 0$ when $1 < p < 2$. Using (4.2), we see that $h_p(x)$ is positive when x is large. Therefore, from the fact that $h_p(x)$ is continuous, it follows that $\mu_p > 0$. □

Lemma 5.2. *If $p < 1$, then $h_p(x) > 0$ on $[0, \infty)$. Moreover $h_1(x) > 0$ on $(0, \infty)$.*

Proof. We first show that $h_p(x) > 0$ on $[0, \infty)$ if $p \leq -\frac{1}{2}$. This is equivalent to $U(-p - \frac{1}{2}, x) > 0$ on $[0, \infty)$ if $p \leq -\frac{1}{2}$.

Recall from §2 that $U(-p - \frac{1}{2}, x)$ satisfies

$$y'' - (\tfrac{1}{4}x^2 - \tfrac{1}{2} - p)y = 0. \tag{5.5}$$

By our condition on p, $\frac{1}{4}x^2 - \frac{1}{2} - p > 0$ for all $x \in \mathbb{R}\backslash\{0\}$. Also, by 19.3.5 of [1],

$$U\left(-p - \frac{1}{2}, 0\right) = \frac{2^{p/2}\sqrt{\pi}}{\Gamma((1-p)/2)} > 0, \tag{5.6}$$

$$U'\left(-p - \frac{1}{2}, 0\right) = -\frac{2^{(p+1)/2}\sqrt{\pi}}{\Gamma(-p/2)} < 0 \tag{5.7}$$

since $-\frac{p}{2} \geq 0$.

Suppose U has a zero in $(0, \infty)$. Let z_1 be its smallest positive zero. Then, by (5.6),

$$U(-p - \tfrac{1}{2}, x) > 0 \quad \text{on } [0, z_1), \tag{5.8}$$

and, by (5.5),

$$U''(-p - \tfrac{1}{2}, x) > 0 \quad \text{on } (0, z_1). \tag{5.9}$$

Case (i). $U'(-p - \frac{1}{2}, z_1) > 0$.

By (5.7), U' has a zero in $(0, z_1)$. Let z_2 be its smallest one. The inequality (5.9) and the mean value theorem of calculus imply

$$U'(-p - \tfrac{1}{2}, x) > 0 \quad \text{on } (z_2, z_1). \tag{5.10}$$

The mean value theorem once more implies, by (5.8) and (5.10), that

$$U(-p - \tfrac{1}{2}, z_1) > 0$$

which is contrary to z_1 being a zero of U.

Case (ii). $U'(-p - \frac{1}{2}, z_1) = 0$.

Since $U(-p - \frac{1}{2}, z_1) = 0$, the uniqueness of the solution of (5.5) implies that $U(-p - \frac{1}{2}, x) \equiv 0$, contrary to (5.6).

Case (iii). $U'(-p - \frac{1}{2}, z_1) < 0$.

Because $U(-p - \frac{1}{2}, z_1) = 0$, the continuity of U' at z_1 and the mean value theorem show that, for some $\varepsilon > 0$,

$$U(-p - \tfrac{1}{2}, x) < 0 \quad \text{on } (z_1, z_1 + \varepsilon).$$

By (4.2), $z_3 = \inf\{x > z_1 \colon U(-p - \frac{1}{2}, x) = 0\} < \infty$, and

$$U(-p - \tfrac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3).$$

Hence, by (5.5),

$$U''(-p - \tfrac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3).$$

Thus by the mean value theorem and $U'(-p - \frac{1}{2}, z_1) < 0$,

$$U'(-p - \tfrac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3). \tag{5.11}$$

However by Rolle's theorem, since $U(-p - \frac{1}{2}, z_1) = U(-p - \frac{1}{2}, z_3) = 0$, there exists a $z_4 \in (z_1, z_3)$, such that

$$U'(-p - \tfrac{1}{2}, z_4) = 0$$

which is contrary to (5.11). Thus the lemma is true when $p \le -\frac{1}{2}$.

For the remaining p's, let us first consider $-\frac{1}{2} < p \le 0$. By (4.3)

$$h_p'(x) = p h_{p-1}(x) \quad \text{on } [0, \infty).$$

Thus, using $h_{p-1}(x) > 0$, the above line, and $p \le 0$, we have

$$h_p'(x) \le 0 \quad \text{on } [0, \infty). \tag{5.12}$$

Suppose there exists a $z_5 \geq 0$ such that $h_p(z_5) \leq 0$. Then, by the mean value theorem and (5.12),

$$h_p(x) \leq 0 \quad \text{on } (z_5, \infty),$$

which is contrary to (4.2). So $h_p(x) > 0$ when $-\frac{1}{2} < p \leq 0$.

Finally when $0 < p \leq 1$, again, by (4.3) we have

$$h_p'(x) > 0 \quad \text{on } (0, \infty). \tag{5.13}$$

Using $h_p(0) = (2^{p/2}/\sqrt{\pi})\Gamma((p+1)/2)\cos(p\pi/2) > 0$ for $0 < p < 1$, $h_1(0) = 0$, and (5.13), we have $h_p(x) > 0$ on $[0, \infty)$ for $0 < p < 1$ and $h_1(x) > 0$ on $(0, \infty)$. This completes the proof. □

Lemma 5.3. (a) *For* $1 \leq p \leq 2$,

$$h_p(x) > 0 \quad on\ (\mu_p, \infty),$$

$$h_p'(x) > 0, \quad h_p''(x) \geq 0, \quad h_p^{(3)}(x) \leq 0 \quad and \quad h_p^{(4)}(x) \geq 0 \quad on\ (0, \infty).$$

(b) *For* $2 < p \leq 3$,

$$h_p(x) > 0 \quad on\ (\mu_p, \infty),$$

$$h_p'(x) > 0 \quad on\ (\mu_{p-1}, \infty),$$

$$h_p''(x) > 0, \quad h_p^{(3)}(x) > 0 \quad and \quad h_p^{(4)}(x) \leq 0 \quad on\ (0, \infty).$$

(c) *For* $p > 3$,

$$\begin{aligned} h_p(x) &> 0 && on\ (\mu_p, \infty), \\ h_p'(x) &> 0 && on\ (\mu_{p-1}, \infty), \\ h_p''(x) &> 0 && on\ (\mu_{p-2}, \infty), \\ h_p^{(3)}(x) &> 0 && on\ (\mu_{p-3}, \infty), \\ h_p^{(4)}(x) &> 0 && on\ (\mu_{p-4}, \infty). \end{aligned}$$

In particular, $h_p^{(n)}(x) > 0$ *on* (μ_p, ∞) *for* $n = 0, 1, 2, 3, 4$.

Notice that by Lemma 5.2, if $p \leq n + 1$, then $\mu_{p-n} = 0$ according to our definition.

Proof. Let us first show that $h_p(x) > 0$ on (μ_p, ∞) for all p. If $p \leq 1$, this follows from Lemma 5.2. Now consider the case $p > 1$. By Lemma 5.1, since μ_p is the largest positive zero of $h_p(x)$, $h_p(x)$ must keep the same sign in (μ_p, ∞). By (4.2),

$$h_p(x) > 0 \quad \text{on } (\mu_p, \infty), \tag{5.15}$$

since $h_p(x) > 0$ when x is large. Thus, using (4.3), we have

$$h_p''(x) = p h_{p-1}'(x) = p(p-1)h_{p-2}(x), \tag{5.16}$$

$$h_p^{(3)}(x) = p(p-1)(p-2)h_{p-3}(x), \tag{5.17}$$

and

$$h_p^{(4)}(x) = p(p-1)(p-2)(p-3)h_{p-4}(x). \tag{5.18}$$

The lemma now follows from (5.15)–(5.18), (4.3), Lemma 5.1, and Lemma 5.2. □

Proof of (5.1). There are two cases.

Case (i). $p \geq 2$. We want to show that

$$w_p'(x) \geq v_p'(x) \quad \text{on } [\mu_p, \infty). \tag{5.19}$$

Then by the mean value theorem and $w_p(\mu_p) = v_p(\mu_p) = 0$, we get

$$w_p(x) \geq v_p(x) \quad \text{on } [\mu_p, \infty). \tag{5.20}$$

Consider $B_p(x) = w_p'(x)/v_p'(x) = \frac{1}{p}\alpha_p h_p'(x)/x^{p-1}$. Then

$$\begin{aligned} B_p'(x) &= \tfrac{1}{p}\alpha_p\{x^{p-1}h_p''(x) - (p-1)x^{p-2}h_p'(x)\}/x^{2p-2} \\ &= \tfrac{1}{p}\alpha_p\{xh_p''(x) - (p-1)h_p'(x)\}/x^p. \end{aligned}$$

Differentiating (4.1) once, we have

$$U^{(3)}(x) - xU''(x) + (p-1)U'(x) = 0.$$

Thus, $h_p^{(3)}(x) - xh_p''(x) + (p-1)h_p'(x) = 0$, or

$$h_p^{(3)}(x) = xh_p''(x) - (p-1)h_p'(x). \tag{5.21}$$

Using (5.21) in the last equality of $B_p'(x)$, (b) and (c) of Lemma 5.3, and the definition of α_p, we have that

$$B_p'(x) = \tfrac{1}{p}\alpha_p h_p^{(3)}(x)/x^p \geq 0 \quad \text{on } (\mu_p, \infty).$$

Hence, the mean value theorem and $B_p(\mu_p) = 1$ imply (5.19).

Case (ii). $1 \leq p \leq 2$. Again let $B_p(x) = w_p'(x)/v_p'(x) = -\frac{1}{p}\alpha_p h_p'(x)/x^{p-1}$. As in Case (i), $B_p'(x) = -\frac{1}{p}\alpha_p h_p^{(3)}(x)/x^p$. Since $\alpha_p = -p\mu_p^{p-1}/h_p'(\mu_p) \leq 0$ and $h_p^{(3)}(x) \leq 0$ on $(0, \infty)$ by Lemma 5.3(a), we have $B_p'(x) \leq 0$ on $(0, \infty)$. Now using the mean value theorem and $B_p(\mu_p) = 1$, we get (5.19) and (5.20), thus (5.1). □

Before we go to the proof of (4.5), we show an important lemma which will be needed very often later.

Lemma 5.4. $\mu_p^2 \geq p-1$ *if* $p \geq 2$ *and* $\mu_p^2 \leq p-1$ *if* $1 \leq p \leq 2$.

Proof. From (4.1) and $h_p(\mu_p) = 0$, we see $h_p''(\mu_p) = \mu_p h_p'(\mu_p)$. Using (5.21) we have

$$h_p^{(3)}(\mu_p) = \mu_p h_p''(\mu_p) - (p-1)h_p'(\mu_p) = (\mu_p^2 - (p-1))h_p'(\mu_p).$$

Applying (5.17) and Lemmas 5.1 and 5.3, we see when $p \geq 2$,

$$\mu_p^2 - (p-1) = h_p^{(3)}(\mu_p)/h_p'(\mu_p) \geq 0,$$

and when $1 \leq p \leq 2$,

$$\mu_p^2 - (p-1) = h_p^{(3)}(\mu_p)/h_p'(\mu_p) \leq 0.$$

This completes the proof. $\square$

6. Proof of the averaging property

We first notice that inequality (4.5) is equivalent to

$$\tag{6.1} \begin{aligned} &\tfrac{1}{2}(t+a^2)^{p/2}\{u_p((x+a)/\sqrt{t+a^2}) + u_p((x-a)/\sqrt{t+a^2})\} \\ &\quad - t^{p/2}u_p(x/\sqrt{t}) \leq 0 \quad \text{for } x \in \mathbb{R},\ a \in \mathbb{R} \text{ and } t > 0. \end{aligned}$$

The case $t = 0$ is from the continuity of U_p.

Without loss of generality we can set $t = 1$. Also, since u_p is an even function, we need to prove (6.1) only for $x \geq 0$ and $a \geq 0$. Thus (6.1) is equivalent to

$$\tag{6.1$'$} \begin{aligned} &\tfrac{1}{2}(1+a^2)^{p/2}\{u_p((x+a)/\sqrt{1+a^2}) + u_p((x-a)/\sqrt{1+a^2})\} \\ &\quad - u_p(x) \leq 0 \quad \text{for } x \geq 0 \text{ and } a \geq 0. \end{aligned}$$

Denote

$$G_x(a) = \tfrac{1}{2}(1+a^2)^{p/2}\{u_p((x+a)/\sqrt{1+a^2}) + u_p((x-a)/\sqrt{1+a^2})\} - u_p(x)$$

and let $y = x/\sqrt{1+a^2}$, $b = a/\sqrt{1+a^2}$. Since $G_x(0) = 0$ for $x \geq 0$, inequality $(6.1)'$ will follow if we can show

$$\tag{6.2} G_x'(a) \leq 0 \quad \text{for all } a \geq 0.$$

In view of the definition of u_p, the proof of (6.2) is conveniently divided into six cases:

Case (I). $0 \leq x \leq \mu_p$, $a \geq 0$, and $(x \pm a)^2 \leq \mu_p^2(1+a^2)$.

Solving the above inequalities, we have

(i) $\sqrt{\mu_p^2 - 1} \leq x \leq \mu_p$ and $0 \leq a \leq \rho_1(x)$ or $a \geq \rho_2(x)$, where

$$\tag{6.3} \rho_1(x) = \frac{x - \sqrt{\mu_p^2(x^2 - \mu_p^2 + 1)}}{\mu_p^2 - 1}, \qquad \rho_2(x) = \frac{x + \sqrt{\mu_p^2(x^2 - \mu_p^2 + 1)}}{\mu_p^2 - 1}$$

or

(ii) $0 \leq x \leq \sqrt{\mu_p^2 - 1}$ and $a \geq 0$.

Differentiating $G_x(a)$, we have

$$\begin{aligned}
G_x'(a) &= \frac{1}{2}pa(1+a^2)^{p/2-1}\left\{\left(\frac{x+a}{\sqrt{1+a^2}}\right)^p - \mu_p^p + \left|\frac{x-a}{\sqrt{1+a^2}}\right|^p - \mu_p^p\right\} \\
&\quad + \frac{1}{2}(1+a^2)^{p/2}\left\{\operatorname{sgn}(a-x)p\left|\frac{x-a}{\sqrt{1+a^2}}\right|^{p-1}\frac{1+ax}{(\sqrt{1+a^2})^3}\right. \\
&\qquad\qquad \left. + p\left(\frac{x+a}{\sqrt{1+a^2}}\right)^{p-1}\frac{1-ax}{(\sqrt{1+a^2})^3}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\left\{p\left|\frac{x-a}{\sqrt{1+a^2}}\right|^{p-1}\left(\frac{a|x-a|}{1+a^2} + \operatorname{sgn}(a-x)\frac{1+ax}{1+a^2}\right)\right. \\
&\qquad\qquad \left. + p\left(\frac{x+a}{\sqrt{1+a^2}}\right)^{p-1}\left(\frac{a(x+a)}{1+a^2} + \frac{1-ax}{1+a^2}\right) - 2\frac{pa\mu_p^p}{\sqrt{1+a^2}}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\left\{p\left(\frac{x+a}{\sqrt{1+a^2}}\right)^{p-1} + \operatorname{sgn}(a-x)p\left|\frac{x-a}{\sqrt{1+a^2}}\right|^{p-1}\right. \\
&\qquad\qquad \left. -2\frac{pa\mu_p^p}{\sqrt{1+a^2}}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\{v_p'(y+b) + \operatorname{sgn}(b-y)v_p'(|y-b|) - 2pb\mu_p^p\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}C_y(b) \quad \text{if } 0 \le b < y \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\mathscr{C}_y(b) \quad \text{if } 0 \le y < b,
\end{aligned}$$

where

$$C_y(b) = v_p'(y+b) - v_p'(y-b) - 2bp\mu_p^p$$

under the condition

$$(6.3.1) \qquad 0 \le b \le y, \qquad y+b \le \mu_p,$$

and

$$\mathscr{C}_y(b) = v_p'(y+b) + v_p'(b-y) - 2bp\mu_p^p$$

under the condition

$$(6.3.2) \qquad 0 \le y \le b, \qquad y+b \le \mu_p.$$

Lemma 6.1. *Both $C_y(b)$ and $\mathscr{C}_y(b)$ are nonpositive on the domain on which they are defined.*

Proof. Assume that (6.3.1) holds and recall that, for $p \geq 3$, we have $v_p(x) = |x|^p - \mu_p^p$. So, if $x \geq 0$, then $v_p''(x) = p(p-1)x^{p-2}$. By (6.3.1) and Lemma 5.4,

$$\begin{aligned} C_y'(b) &= v_p''(y+b) + v_p''(y-b) - 2p\mu_p^p \\ &= p(p-1)\{(y+b)^{p-2} + (y-b)^{p-2}\} - 2p\mu_p^p \\ &\leq 2p(p-1)\mu_p^{p-2} - 2p\mu_p^p \\ &= 2p\mu_p^{p-2}((p-1) - \mu_p^2) \leq 0. \end{aligned}$$

This, together with $C_y(0) = 0$, implies that $C_y(b) \leq 0$ on $[0, \mu_p - y]$.

Now assume that (6.3.2) holds. Then $\mathscr{C}_y'(b) \leq 0$ on $[y, \mu_p - y]$ by an argument similar to the one above. By Lemma 5.4 and $2y \leq \mu_p$,

$$\begin{aligned} \mathscr{C}_y(y) &= v_p'(2y) - 2yp\mu_p^p \\ &= p(2y)^{p-1} - 2yp\mu_p^p \\ &\leq 2yp\mu_p^{p-2}(1 - \mu_p^2) \leq 0. \end{aligned}$$

So $\mathscr{C}_y(b) \leq 0$ on $[y, \mu_p - y]$. This completes the proof of Lemma 6.1. $\square$

Thus (6.2) is proven by Lemma 6.1 under Case (I).

Case (II). $0 \leq x \leq \mu_p$, $a \geq 0$, and $(x+a)^2 \geq \mu_p^2(1+a^2)$, $(x-a)^2 \leq \mu_p^2(1+a^2)$.

Solving the above inequalities, we have

$$\text{(6.4)} \qquad \begin{aligned} &\text{(i) } \rho_1(x) \leq a \leq \rho_2(x), \text{ if } \sqrt{\mu_p^2 - 1} \leq x \leq \mu_p, \\ &\text{(ii) No solution if } x < \sqrt{\mu_p^2 - 1}, \end{aligned}$$

where $\rho_1(x)$ and $\rho_2(x)$ are defined in (6.3).

Notice that, by Lemma 5.1 and our assumption that $0 \leq x \leq \mu_p$,

$$\begin{aligned} p \geq 3 &\Rightarrow \mu_p^2 \geq \mu_3^2 = 3 \\ &\Rightarrow \mu_p^2 - 4 \geq -1 \\ &\Rightarrow (\mu_p^2 - 1)\{x^2(\mu_p^2 - 4) + \mu_p^2\} \geq 0 \\ &\Rightarrow (\mu_p^2 - 2)^2 x^2 \geq \mu_p^2(x^2 - \mu_p^2 + 1) \\ &\Rightarrow x \geq \rho_2(x). \end{aligned}$$

So $\rho_1(x) \leq \rho_2(x) \leq x$. Thus, by (6.4), $x \geq a$.

Using (4.1),

$$
\begin{aligned}
G_x'(a) &= \frac{1}{2}pa(1+a^2)^{p/2-1}\left\{\alpha_p h_p\left(\frac{x+a}{\sqrt{1+a^2}}\right) + v_p\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right\} \\
&\quad + \frac{1}{2}(1+a^2)^{p/2}\left\{\alpha_p h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{(\sqrt{1+a^2})^3}\right. \\
&\qquad\qquad \left. -v_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\frac{1+ax}{(\sqrt{1+a^2})^3}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\left\{\frac{pa\alpha_p}{\sqrt{1+a^2}}h_p\left(\frac{x+a}{\sqrt{1+a^2}}\right) + \frac{pa}{\sqrt{1+a^2}}v_p\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right. \\
&\qquad \left. + \alpha_p h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{1+a^2} - v_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\frac{1+ax}{1+a^2}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\left\{\frac{a\alpha_p}{\sqrt{1+a^2}}\left[-h_p''\left(\frac{x+a}{\sqrt{1+a^2}}\right)\right.\right. \\
&\qquad \left. + \frac{x+a}{\sqrt{1+a^2}}h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\right] \\
&\qquad + \alpha_p h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{1+a^2} - \frac{pa\mu_p^p}{\sqrt{1+a^2}} \\
&\qquad \left. + p\left(\frac{x-a}{\sqrt{1+a^2}}\right)^{p-1}\left[\frac{a}{\sqrt{1+a^2}}\frac{x-a}{\sqrt{1+a^2}} - \frac{1+ax}{1+a^2}\right]\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\left\{\alpha_p h_p'\left(\frac{x+\alpha}{\sqrt{1+a^2}}\right) - v_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right. \\
&\qquad \left. - \frac{a}{\sqrt{1+a^2}}\left(\alpha_p h_p''\left(\frac{x+a}{\sqrt{1+a^2}}\right) + p\mu_p^p\right)\right\}.
\end{aligned}
$$

We will show $G_x'(a) \le 0$ on $[\rho_1(x), \rho_2(x)]$ by the following lemma.

Lemma 6.2. *Define*

$$D_y(b) = \alpha_p h_p'(y+b) - v_p'(y-b) - \alpha_p b h_p''(y+b) - pb\mu_p^p$$

on the union of

(6.4.1) $$0 \le y \le \mu_p \le 2y, \qquad \mu_p - y \le b \le y,$$

and

(6.4.2) $$\mu_p \le y, \qquad y - \mu_p \le b \le y;$$

$$\mathscr{D}_y(b) = \alpha_p h_p'(y+b) + v_p'(b-y) - \alpha_p b h_p''(y+b) - pb\mu_p^p$$

on the union of

(6.4.3) $$\mu_p \le 2y, \qquad y \le b \le y + \mu_p,$$

and

$$(6.4.4) \qquad 2y \le \mu_p, \qquad \mu_p - y \le b \le y + \mu_p.$$

Then both $D_y(b)$ and $\mathscr{D}_y(b)$ are nonpositive on their respective domains.

Before we start the proof, we observe that the union of (6.4.1) and (6.4.2) is the set

$$(6.4.5) \qquad 0 \le b \le y, \quad y + b \ge \mu_p, \quad \text{and} \quad y - b \le \mu_p.$$

The union of (6.4.3) and (6.4.4) is the set

$$(6.4.6) \qquad 0 \le y \le b, \quad y + b \ge \mu_p, \quad \text{and} \quad b - y \le \mu_p.$$

Proof. Under (6.4.5), by Lemmas 5.3 and 5.4, we have

$$(6.4.7) \qquad \begin{aligned} D_y'(b) &= -\alpha_p b h_p^{(3)}(y+b) + v_p''(y-b) - p\mu_p^p \\ &\le -\alpha_p b h_p^{(3)}(y+b) + p(p-1)\mu_p^{p-2} - p\mu_p^p \\ &\le p\mu_p^{p-2}((p-1) - \mu_p^2) \le 0. \end{aligned}$$

Similarly, under (6.4.6),

$$(6.4.8) \qquad \mathscr{D}_y'(b) = -\alpha_p b h_p^{(3)}(y+b) + v_p''(b-y) - p\mu_p^p \le 0.$$

Assume (6.4.1) and fix a $y \in [\mu_p/2, \mu_p]$,

$$D_y(\mu_p - y) = \alpha_p h_p'(\mu_p) - v_p'(\mu_p - 2b) - \alpha_p b h_p''(\mu_p) - pb\mu_p^p,$$

where $0 \le b = \mu_p - y \le \mu_p$. For $b \in [0, \mu_p]$, let

$$d_1(b) = \alpha_p h_p'(\mu_p) - v_p'(\mu_p - 2b) - \alpha_p b h_p''(\mu_p) - pb\mu_p^p.$$

Then by Lemma 5.4 and the fact that $|v_p''(x)| \le p(p-1)|x|^{p-2}$, $|\mu_p - 2b| \le \mu_p$, and $p\mu_p^p = \alpha_p h_p''(\mu_p)$, we have on $[0, \mu_p]$,

$$\begin{aligned} d_1(0) &= \alpha_p h_p'(\mu_p) - v_p'(\mu_p) = 0, \\ d_1'(b) &= 2v_p''(\mu_p - 2b) - p\mu_p^p - \alpha_p h_p''(\mu_p) \\ &\le 2p(p-1)|\mu_p - 2b|^{p-2} - p\mu_p^p - \alpha_p h_p''(\mu_p) \\ &\le 2p(p-1)\mu_p^{p-2} - 2p\mu_p^p = 2p\mu_p^{p-2}((p-1) - \mu_p^2) \le 0. \end{aligned}$$

Thus by the mean value theorem, $d_1(b) \le 0$ on $[0, \mu_p]$. Consequently, $d_1(\mu_p - y) = D_y(\mu_p - y) \le 0$. Hence by (6.4.7), $D_y(b) \le 0$ on $[\mu_p - y, y]$.

Assume (6.4.2) and fix a $y \in [\mu_p, \infty)$; then

$$D_y(y - \mu_p) = \alpha_p h_p'(\mu_p + 2b) - v_p'(\mu_p) - \alpha_p b h_p''(\mu_p + 2b) - pb\mu_p^p,$$

where $0 \le b = y - \mu_p$. For $b \in [0, \infty)$, let

$$d_2(b) = \alpha_p h_p'(\mu_p + 2b) - v_p'(\mu_p) - \alpha_p b h_p''(\mu_p + 2b) - pb\mu_p^p.$$

Then

$$d_2(0) = \alpha_p h_p'(\mu_p) - v_p'(\mu_p) = 0,$$
$$d_2'(b) = \alpha_p h_p''(\mu_p + 2b) - 2\alpha_p b h_p^{(3)}(\mu_p + 2b) - p\mu_p^p.$$

So, by $\alpha_p h_p''(\mu_p) = p\mu_p^p$ and Lemma 5.3,

$$d_2'(0) = \alpha_p h_p''(\mu_p) - p\mu_p^p = 0,$$
$$d_2''(b) = -4\alpha_p b h_p^{(4)}(\mu_p + 2b) \le 0.$$

Thus on $[0, \infty)$, $d_2(b) \le 0$ by the mean value theorem. Consequently,

$$D_y(y - \mu_p) = d_2(y - \mu_p) \le 0.$$

Using (6.4.7), we have $D_y(b) \le 0$ on $[y - \mu_p, y]$.

Assume (6.4.3), and fix $y \in [\mu_p/2, \mu_p]$,

$$\mathcal{D}_y(y) = \alpha_p h_p'(2y) - \alpha_p y h_p''(2y) - py\mu_p^p.$$

For $y \in [\mu_p/2, \infty)$, let

$$\mathcal{d}_1(y) = \alpha_p h_p'(2y) - \alpha_p y h_p''(2y) - py\mu_p^p.$$

Then, by essentially the same computation as above, we get

$$\mathcal{d}_1(\mu_p/2) = p\mu_p^{p-1}(1 - \mu_p^2) \le 0,$$
$$\mathcal{d}_1'(\mu_p/2) = -\mu_p \alpha_p h_p^{(3)}(\mu_p) \le 0$$

and on $[\mu_p/2, \infty)$,

$$\mathcal{d}_1''(y) \le 0.$$

Now using the mean value theorem, we have for $y \in [\mu_p/2, \infty)$, $\mathcal{d}_1'(y) \le 0$ and $\mathcal{d}_1(y) \le 0$. Thus (6.4.8) and $\mathcal{D}_y(y) = \mathcal{d}_1(y)$ imply that

$$\mathcal{D}_y(b) \le 0 \quad \text{on } [y, \mu_p + y].$$

Finally assume (6.4.4) and fix a $y \in [0, \mu_p/2]$; then

$$\mathcal{D}_y(\mu_p - y) = \alpha_p h_p'(\mu_p) + v_p'(2b - \mu_p) - \alpha_p b h_p''(\mu_p) - pb\mu_p^p,$$

where $\mu_p/2 \le b = \mu_p - y \le \mu_p$. For $b \in [\mu_p/2, \mu_p]$, let

$$\mathcal{d}_2(b) = \alpha_p h_p'(\mu_p) + v_p'(2b - \mu_p) - \alpha_p b h_p''(\mu_p) - pb\mu_p^p.$$

Then again,

$$\mathcal{d}_2(\mu_p/2) = p\mu_p^{p-1}(1 - \mu_p^2) \le 0$$

and

$$\mathcal{d}_2'(b) = 2p(p-1)(2b - \mu_p)^{p-2} - 2p\mu_p^p \le 0.$$

Thus by the mean value theorem, $\mathscr{d}_2(b) \leq 0$ on $[\mu_p/2, \mu_p]$. So $\mathscr{D}_y(\mu_p - y) = \mathscr{d}_2(\mu_p - y) \leq 0$, and by (6.4.8)

$$\mathscr{D}_y(b) \leq 0 \quad \text{on } [\mu_p - y, \mu_p + y].$$

This finishes the proof. □

Now back to $G'_x(a)$. Since

$$G'_x(a) = \tfrac{1}{2}(1 + a^2)^{(p-1)/2} D_y(b),$$

so by Lemma 6.2

$$G'_x(a) \leq 0 \quad \text{on } [\rho_1(x), \rho_2(x)].$$

That is, (6.2) is proven in this case.

Case (III). $0 \leq x < \mu_p$, $a \geq 0$, and $(x \pm a)^2 \geq \mu_p^2(1 + a^2)$.

We will show this is an impossible case.

Since $0 \leq x < \mu_p$, $a \geq 0$, and, by Lemma 5.4, $\mu_p \geq 1$ for $p \geq 3$,

$$(x - a)^2 = x^2 - 2ax + a^2 \leq x^2 + a^2 < \mu_p^2 + \mu_p^2 a^2 = \mu_p^2(1 + a^2)$$

which is contrary to $(x - a)^2 \geq \mu_p^2(1 + a^2)$.

Case (IV). $x \geq \mu_p$, $a \geq 0$, and $(x \pm a)^2 \leq \mu_p^2(1 + a^2)$.

As in Case (I),

$$\begin{aligned} G'_x(a) &= \tfrac{1}{2}(1 + a^2)^{(p-1)/2} C_y(b), \quad \text{if } 0 \leq a < x, \\ &= \tfrac{1}{2}(1 + a^2)^{(p-1)/2} \mathscr{C}_y(b), \quad \text{if } 0 \leq x < a. \end{aligned}$$

Thus $G'_x(a) \leq 0$ by Lemma 6.1 in this case.

Case (V). $x \geq \mu_p$, $a \geq 0$, and $(x + a)^2 \geq \mu_p^2(1 + a^2)$, $(x - a)^2 \leq \mu_p^2(1 + a^2)$.

As in Case (III),

$$\begin{aligned} G'_x(a) &= \tfrac{1}{2}(1 + a^2)^{(p-1)/2} D_y(b), \quad \text{if } 0 \leq a < x, \\ &= \tfrac{1}{2}(1 + a^2)^{(p-1)/2} \mathscr{D}_y(b), \quad \text{if } 0 \leq x < a. \end{aligned}$$

Thus $G'_x(a) \leq 0$ by Lemma 6.2 in this case.

Case (VI). $x \geq \mu_p$, $a \geq 0$, and $(x \pm a)^2 \geq \mu_p^2(1 + a^2)$.

Solving the above inequalities, we have

$$(6.5) \qquad 0 \leq a \leq -\rho_1(x)$$

where $\rho_1(x)$ is defined in (6.3). Using the condition $x \geq \mu_p$ and Lemma 5.1, we see

$$\begin{aligned} p \geq 2 &\Rightarrow \mu_p^2 \geq 1 \\ &\Rightarrow (x^2 + 1)(\mu_p^2 - 1) \geq 0 \\ &\Rightarrow x\mu_p^2 \geq \sqrt{\mu_p^2(x^2 - \mu_p^2 + 1)} \\ &\Rightarrow x \geq -\rho_1(x). \end{aligned}$$

So when $2 \le p$, $-\rho_1(x) \le x$. Hence for $0 \le a \le -\rho_1(x)$, by (4.1),

$$
\begin{aligned}
G_x'(a) &= \frac{1}{2} pa(1+a^2)^{p/2-1}\alpha_p \left\{ h_p\left(\frac{x+a}{\sqrt{1+a^2}}\right) + h_p\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right\} \\
&\quad + \frac{1}{2}\alpha_p(1+a^2)^{p/2}\left\{ h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{(\sqrt{1+a^2})^3}\right. \\
&\qquad\qquad \left. - h_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\frac{1+ax}{(\sqrt{1+a^2})^3}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\alpha_p\left\{\frac{pa}{\sqrt{1+a^2}}h_p\left(\frac{x+a}{\sqrt{1+a^2}}\right) + \frac{pa}{\sqrt{1+a^2}}h_p\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right. \\
&\qquad\qquad \left. + h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{1+a^2} - h_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\frac{1+ax}{1+a^2}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\alpha_p\left\{\frac{a}{\sqrt{1+a^2}}\left[-h_p''\left(\frac{x+a}{\sqrt{1+a^2}}\right)\right.\right. \\
&\qquad\qquad + \frac{x+a}{\sqrt{1+a^2}}h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right) \\
&\qquad\qquad \left. - h_p''\left(\frac{x-a}{\sqrt{1+a^2}}\right) + \frac{x-a}{\sqrt{1+a^2}}h_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right] \\
&\qquad\qquad \left. + h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right)\frac{1-ax}{1+a^2} - h_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\frac{1+ax}{1+a^2}\right\} \\
&= \frac{1}{2}(1+a^2)^{(p-1)/2}\alpha_p\left\{h_p'\left(\frac{x+a}{\sqrt{1+a^2}}\right) - h_p'\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right. \\
&\qquad\qquad \left. - \frac{a}{\sqrt{1+a^2}}\left[h_p''\left(\frac{x+a}{\sqrt{1+a^2}}\right) + h_p''\left(\frac{x-a}{\sqrt{1+a^2}}\right)\right]\right\}.
\end{aligned}
$$

Define

(6.5.1) $$E_y(b) = \alpha_p\{h_p'(y+b) - h_p'(y-b) - b[h_p''(y+b) + h_p''(y-b)]\}$$

on $\mu_p \le y$ and $b \in [0, y-\mu_p]$.

Note that (6.5.1) is equivalent to

(6.5.2) $$0 \le b \le y, \qquad \mu_p \le y+b, \qquad \mu_p \le y-b.$$

We will show $G_x'(a) \le 0$ on $[0, -\rho_1(x)]$ by the following lemma.

Lemma 6.3. (a) $E_y(b)$ *is nonpositive on the domain it is defined when* $p \ge 3$.

(b) $E_y(b) \ge 0$ *on the domain it is defined when* $1 < p < 2$ *and* $2 < p < 3$.

Proof. (a) Assume (6.5.1) and fix $y \ge \mu_p$. By (6.5.2), Lemma 5.3, and the mean value theorem,

$$E_y'(b) = \alpha_p[-bh_p^{(3)}(y+b) + bh_p^{(3)}(y-b)] = -2b^2\alpha_p h_p^{(4)}(\xi) \le 0,$$

where $\mu_p \le \xi \in (y-b, y+b)$. Hence by the mean value theorem and $E_y(0) = 0$,

$$E_y(b) \le 0 \quad \text{on } [0, y-\mu_p].$$

(b) By the mean value theorem, there exists a $\zeta \in (y-b, y+b)$ such that

$$E_y'(b) = -2b^2\alpha_p h_p^{(4)}(\zeta).$$

From (6.5.2), $\mu_p \le \zeta$. When $2 < p < 3$, Lemma 5.3(b) and the expression α_p imply that

$$\alpha_p > 0 \quad \text{and} \quad E_y'(b) > 0 \quad \text{on } [0, y-\mu_p].$$

So the mean value theorem and $E_y(0) = 0$ give

$$E_y(b) \ge 0 \quad \text{on } [0, y-\mu_p].$$

When $1 < p < 2$, Lemma 5.3(a) implies

$$\alpha_p < 0 \quad \text{and} \quad E_y'(b) > 0 \quad \text{on } [0, y-\mu_p].$$

So again we have

$$E_y(b) \ge 0 \quad \text{on } [0, y-\mu_p].$$

This concludes the proof. □

We turn again to $G_x'(a)$. Since

$$G_x'(a) = \tfrac{1}{2}(1+a^2)^{(p-1)/2}E_y(b),$$

So by Lemma 6.3 $G_x'(a) \le 0$ on $[0, -\rho_1(x)]$. This completes the proof of (6.2).

7. Proof of the negativity property

From the definition of $U_p(x)$, (4.6) is equivalent to

(7.1) $$u_p(1) \le 0 \quad \text{for } p > 1.$$

By the definition of $u_p(x)$ and Lemma 5.4, (7.1) is equivalent to

(7.2) $$v_p(1) \le 0 \quad \text{if } p \ge 2,$$

and

(7.3) $$\alpha_p h_p(1) \le 0 \quad \text{if } 1 < p \le 2.$$

Now using Lemma 5.3(a) and Lemma 5.4, we have $v_p(1) \le 0$ for $p \ge 2$, and $\alpha_p \le 0$ and $h_p(1) \ge 0$ for $1 < p < 2$. So (7.2) and (7.3) both hold.

8. Remark on the case $p < 3$

We discuss here why (4.5) does not hold for $1 < p < 2$ and $2 < p < 3$. In fact we will show $(6.1)'$ fails to be true when $x \ge \mu_p$, $a \ge 0$, and $(x \pm a)^2 \ge \mu_p^2(1+a^2)$ in both cases. When $p = 1$ or $p = 2$, (4.5) is trivially true.

If $1 < p < 2$, solving $x \geq \mu_p$, $(x \pm a)^2 \geq \mu_p^2(1+a^2)$, we have

$$0 \leq a \leq -\rho_1(x), \quad \text{or} \quad a \geq -\rho_2(x),$$

where $\rho_1(x)$ and $\rho_2(x)$ are defined in (6.3).

A similar argument as in Case (VI) shows that $x \leq -\rho_1(x)$ when $1 < p < 2$. Fix an $x \geq \mu_p$. For $a \in (0, x]$, let $y = x/\sqrt{1+a^2}$ and $b = a/\sqrt{1+a^2}$, then y and b satisfy (6.5.2). Hence, by the proof of Lemma 6.3(b),

$$G_x'(a) = \tfrac{1}{2}(1+a^2)^{(p-1)/2} E_y(b) > 0 \quad \text{on } (0, x].$$

Thus, by the mean value theorem and $G_x(0) = 0$,

$$G_x(a) > 0 \quad \text{on } (0, x],$$

which is contrary to $(6.1)'$.

If $2 < p < 3$, then $\mu_p \leq x$, $a \geq 0$, $(x \pm a)^2 \geq \mu_p^2(1+a^2)$ imply that $0 \leq a \leq -\rho_1(x)$, where $\rho_1(x)$ is defined in (6.3), and $x \geq -\rho_1(x)$ as we have shown in Case (VI). Fix $x \geq \mu_p$. Thus $y = x/\sqrt{1+a^2}$ and $b = a/\sqrt{1+a^2}$ satisfy (6.5.3); we then have on $(0, -\rho_1(x)]$, by Lemma 6.3(b),

$$G_x'(a) = \tfrac{1}{2}(1+a^2)^{(p-1)/2} E_y(b) > 0.$$

So $G_x(a) > 0$ on $(0, -\rho_1(x)]$ by the mean value theorem and $G_x(0) = 0$, which is contrary to $(6.1)'$.

9. The existence of the function $U_p(x, t)$: Hilbert space case

We want to generalize the above results to Hilbert spaces. Namely, we want to show that the analogues of (4.4)–(4.6) hold in Hilbert spaces. We will then have $(1.7)'$ of Theorem $1'$.

Let α_p and u_p be as in §4. Define for $x \in \mathbb{H}$, $t \geq 0$,

$$\begin{aligned} U_p(x, t) &= t^{p/2} u_p(|x|/\sqrt{t}) \quad \text{if } t > 0, \\ &= \alpha_p |x|^p \quad \text{if } t = 0, \end{aligned}$$

for any inner product spaces $\mathbb{H}$. For any element $x \in \mathbb{H}$, denote $X = |x|$, $\cos\theta = (x \cdot a)/XA$, where $\cdot$ is the inner product. Then

$$|x+a| = \sqrt{X^2 + A^2 + 2XA\cos\theta}.$$

The analogue of (4.4) is

$$U_p(x, t) \geq V_p(x, t). \tag{9.1}$$

This is trivially true by the definition and (4.4).

The analogue of (4.5) is

$$\tfrac{1}{2}\{U_p(x+a,\, t+A^2) + U_p(x-a,\, t+A^2)\} - U_p(x,\, t) \le 0 \quad \text{for } p \ge 3 \text{ and all } x \in \mathbb{H},\ a \in \mathbb{H},\ t \ge 0. \tag{9.2}$$

By our definition it is equivalent to

$$\begin{aligned} &\frac{1}{2}(1+A^2)^{p/2}\left\{u_p\left(\left(\frac{X^2+A^2+2AX\cos\theta}{1+A^2}\right)^{1/2}\right)\right. \\ &\qquad\left. + u_p\left(\left(\frac{X^2+A^2-2AX\cos\theta}{1+A^2}\right)^{1/2}\right)\right\} - u_p(X) \\ &\le 0 \quad \text{for } X \ge 0,\ A \ge 0, \text{ and } \cos\theta \in [0,\, 1]. \end{aligned} \tag{9.3}$$

Define

$$\begin{aligned} \mathfrak{K}_p(t) = \frac{1}{2}(1+A^2)^{p/2}&\left\{u_p\left(\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{1/2}\right)\right. \\ &\left. + u_p\left(\left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{1/2}\right)\right\} - u_p(X). \end{aligned}$$

Since $\mathfrak{K}_p(1) = G_X(A) \le 0$, it suffices to show $\mathfrak{K}_p'(t) \ge 0$ on $[0,\, 1]$.

As in the real case, there are six cases according to the definition of $u_p(x)$.

Case (I). $0 \le X \le \mu_p$, $A \ge 0$, $X^2 + A^2 \pm 2AXt \le \mu_p^2(1+A^2)$.

Since

$$X^2 + A^2 + 2AXt \ge X^2 + A^2 - 2AXt$$

and $p \ge 2$, we have

$$\begin{aligned} \mathfrak{K}_p'(t) = \frac{1}{2}pXA(1+A^2)^{p/2-1}&\left\{\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{p/2-1}\right. \\ &\left. - \left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{p/2-1}\right\} \ge 0 \end{aligned}$$

in this case.

Case (II). $0 \le X < \mu_p$, $A \ge 0$, $X^2 + A^2 + 2AXt \ge \mu_p^2(1+A^2)$, and $X^2 + A^2 - 2AXt \le \mu_p^2(1+A^2)$.

In this case,

$$\mathfrak{K}_p'(t) = \frac{1}{2}AX(1+A^2)^{p/2-1}\left\{\alpha_p\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{-1/2}\right.$$
$$\times h_p'\left(\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{1/2}\right)$$
$$\left.-p\left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{p/2-1}\right\}$$
$$\geq \frac{1}{2}AX(1+A^2)^{p/2-1}\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{-1/2}$$
$$\times\left\{\alpha_p h_p'\left(\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{1/2}\right)-p\left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{(p-1)/2}\right\}$$
$$\geq 0$$

since $pX^{p-1} \leq \alpha_p h_p'(X)$ on $[\mu_p, \infty)$ by (5.19).

Case (III). $0 \leq X < \mu_p$, $A \geq 0$, $X^2+A^2 \pm 2AX\cos\theta \geq \mu_p^2(1+A^2)$.

As in the real case in §6, this is impossible.

Case (IV). $X \geq \mu_p$, $A \geq 0$, $X^2+A^2 \pm 2AXt \leq \mu_p^2(1+A^2)$.

Case (V). $X \geq \mu_p$, $A \geq 0$, $X^2+A^2+2AXt \geq \mu_p^2(1+A^2)$, and $X^2+A^2-2AXt \leq \mu_p^2(1+A^2)$.

$\mathfrak{K}_p'(t)$ in Cases (IV) and (V) has the same expression as in Cases (I) and (II). So it is nonnegative.

Case (VI). $X \geq \mu_p$, $A \geq 0$, $X^2+A^2 \pm 2AXt \geq \mu_p^2(1+A^2)$.

In this case,

$$\mathfrak{K}_p'(t) = \frac{1}{2}AX\alpha_p(1+A^2)^{p/2-1}$$
$$\times\left\{\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{-1/2} h_p'\left(\left(\frac{X^2+A^2+2AXt}{1+A^2}\right)^{1/2}\right)\right.$$
$$\left.-\left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{-1/2} h_p'\left(\left(\frac{X^2+A^2-2AXt}{1+A^2}\right)^{1/2}\right)\right\}.$$

Let $C(X) = h_p'(X)/X$ for $X \geq \mu_p$. Then,

$$C'(X) = (Xh_p''(X) - h_p'(X))/X^2$$
$$= (h_p^{(3)}(X) + (p-2)h_p'(X))/X^2 \geq 0$$

by Lemma 5.3 and the fact that h_p satisfies differential equation (4.1). Thus if $\mu_p \le X_1 \le X_2$, then by the mean value theorem, $C(X_1) \le C(X_2)$. Hence if we let

$$X_1 = \left(\frac{X^2 + A^2 - 2AXt}{1 + A^2}\right)^{1/2}, \qquad X_2 = \left(\frac{X^2 + A^2 + 2AXt}{1 + A^2}\right)^{1/2},$$

then, by the above inequality, $\mathfrak{K}_p'(t) \ge 0$.

This completes the proof of (9.2).

The analogue of (4.6) is

$$U_p(a, |a|^2) \le 0 \quad \text{for } a \in \mathbb{H}. \tag{9.12}$$

This is trivially true. Combining (9.1), (9.2), and (9.12), we see that the function $U_p(x, t)$ satisfies the properties described in §3.

10. The existence of the function $\overline{U}_p(x, t)$

Recall from §2 that ν_p is the smallest positive zero of M_p. We define

$$\overline{\alpha}_p = \overline{v}_p'(\nu_p)/M_p'(\nu_p) \quad \text{and} \quad \overline{w}_p(x) = \overline{\alpha}_p M_p(x).$$

Also let

$$\begin{aligned} \overline{u}_p(x) &= \overline{w}_p(|x|) \quad \text{for } 0 \le |x| \le \nu_p, \\ &= \overline{v}_p(|x|) \quad \text{for } \nu_p \le |x| < \infty, \\ \overline{W}_p(x, t) &= t^{p/2}\overline{w}_p(|x|/\sqrt{t}) \quad \text{for } t \ne 0, \end{aligned}$$

and

$$\begin{aligned} \overline{U}_p(x, t) &= t^{p/2}\overline{u}_p(|x|/\sqrt{t}) \qquad \text{for } t > 0, \\ &= -\operatorname{sgn}(p-2)|x|^p \quad \text{for } t = 0. \end{aligned}$$

Then functions $\overline{U}_p(x, t)$ when $p \ge 2$ and $\overline{W}_p(x, t)$ when $0 < p < 2$ satisfy conditions $(3.5)'$–$(3.7)'$ in §3. See Chapter 3 of Wang [17] for details. Thus the existence of the function $\overline{U}_p(x, t)$ is ensured. When $\mathbb{H} = \mathbb{R}$, this gives another proof of Davis' [7] results.

11. The sharpness of the constants ν_p and μ_p

For the case $\mathbb{H} = \mathbb{R}$, Davis [7] showed that ν_p is the best possible constant in (1.6) and (1.8) of Theorem 1. The same procedure can be used to obtain a similar result for μ_p in the case $\mathbb{H} = \mathbb{R}$ and $p \ge 3$. The fact that (1.7) does hold in the real case is the main result of this paper. These inequalities, the inequalities (1.6), (1.7), and (1.8) of Theorem 1, which we have shown to be valid for any Hilbert space $\mathbb{H}$, must therefore be sharp for $\mathbb{H}$.

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